Normals to Level Curves and Tangents

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Overview

We define the tangent plane at a point on a smooth surface in space.

We calculate an equation of the tangent plane from the partial derivatives of the function defining the surface.

This idea is similar to the definition of the tangent line at a point on a curve in the coordinate plane for single-variable functions.

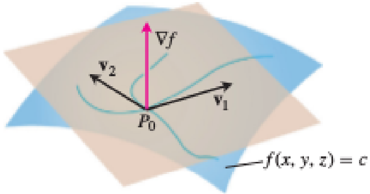
Tangent Planes and Normal Lines

If $\boldsymbol{r} = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f, then $f(g(t), h(t), k(t)) = c$ of this equation with respect to t leads to

$$\begin{aligned}
\frac{d}{dt} f(g(t), h(t), k(t)) &= \frac{d}{dt}(c) \\
\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + + \frac{\partial f}{\partial z}\frac{dk}{dt} &= 0 \\
\left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\right).\left(\frac{dg}{dt}\boldsymbol{i} + \frac{dh}{dt}\boldsymbol{j} + \frac{dk}{dt}\boldsymbol{k}\right) &= 0 \\
\nabla f . \frac{d\boldsymbol{r}}{dt} &= 0.
\end{aligned}$$

At every point along the curve, ∇f is orthogonal to the curve's velocity vector.

Tangent Planes and Normal Lines



Now let us restrict our attention to the curves that pass through P_0. All the velocity vectors at P_0 are orthogonal to ∇f at P_0, so the curve's tangent lines all lie in the plane through P_0 normal to ∇f.

We call this plane the **tangent plane** of the surface at P_0. The line through P_0 perpendicular to the plane is the surface's normal line at P_0.

Tangent Planes and Normal Lines

Definition 1.

The tangent plane *at the point* $P_0(x_0, y_0, z_0)$ *on the level surface* $f(x, y, z) = c$ *of a differentiable function* f *is a plane through* P_0 *normal to* $\nabla f|_{P_0}$.
The normal line *of the surface at* P_0 *is the line through* P_0 *parallel to* $\nabla f|_{P_0}$.

Thus, the tangent plane and normal line have the following equations :

Tangent Plane to $f(x, y, z) = c$ at $P_0 = (x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$

Normal Line to $f(x, y, z) = c$ at $P_0 = (x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t.$$

Tangent Plane and Normal Line : An Example

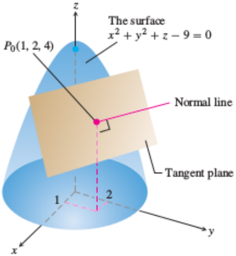
Example 2.

Find the tangent plane and normal line to the surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0$$

at the point $P_0(1, 2, 4)$.

The surface is shown in the figure.



Solution

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0. The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x-1) + 4(y-2) + (z-4) = 0, \quad \text{or} \quad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t.$$

Plane Tangent to a Smooth Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

To find an equation for the plane tangent to a smooth surface $z = f(x, y)$ at a point $P_0(x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$, we first observe that the equation $z = f(x, y)$ is equivalent to $f(x, y) - z = 0$.

The surface $z = f(x, y)$ is therefore the zero level surface of the function

$$F(x, y, z) = f(x, y) - z.$$

The partial derivatives of F are $F_x = f_x,\ F_y = f_y,\ F_z = -1$.

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Example 3.

Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$.

Solution : We calculate the partial derivatives of $f(x, y) = x\cos y - ye^x$ and use Equation(4):

$$f_x(0, 0) = (\cos y - ye^x)_{(0,0)} = 1 - 0 \cdot 1 = 1$$
$$f_x(0, 0) = (-x\sin y - e^x)_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0,$$

or

$$x - y - z = 0.$$

Tangent Line to the Curve of Intersection of Two Surfaces

Let

$$f(x, y, z) = c \quad \text{and} \quad g(x, y, z) = d$$

be two surfaces and let C be the curve of intersection of the surfaces.

The tangent line to C at $P_0(x_0, y_0, z_0)$ is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to

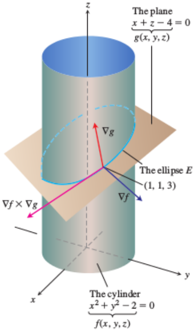
$$\boldsymbol{v} = \nabla f \times \nabla g = v_1 \boldsymbol{i} + v_2 \boldsymbol{j} + v_3 \boldsymbol{k}.$$

Hence the parametric equations of the tangent line to C at $P_0(x_0, y_0, z_0)$ is

$$x = x_0 + v_1 t, \quad y = y_0 + v_2 t, \quad z = z_0 + v_3 t.$$

Example 4.

The surfaces $f(x, y, z) = x^2 + y^2 - 2 = 0$ *and* $g(x, y, z) = x + z - 4 = 0$ *meet in an ellipse* E*. Find parametric equations for the line tangent to* E *at the point* $P_0(1, 1, 3)$.



Solution

The tangent line is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and the coordinates of P_0 give us equations for the line. We have

$$\begin{aligned}
\nabla f|_{(1,1,3)} &= (2x\mathbf{i} + 2y\mathbf{j})_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j} \\
\nabla g|_{(1,1,3)} &= (\mathbf{i} + \mathbf{k})_{(1,1,3)} = \mathbf{i} + \mathbf{k} \\
\mathbf{v} &= (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.
\end{aligned}$$

The tangent line is

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t.$$

Exercise 5.

Find equations for the

(a) *tangent plane and*

(b) *normal line at the point P_0 on the given surface.*

1. $x^2 + y^2 + z^2 = 3, \quad P_0(1, 1, 1)$
2. $2z - x^2 = 0, \quad P_0(2, 0, 2)$
3. $\cos \pi x - x^2 y + e^{xz} + yz = 4, \quad P_0(0, 1, 2)$
4. $x + y + z = 1, \quad P_0(0, 1, 0)$
5. $x^2 + y^2 - 2xy - x + 3y - z = -4, \quad P_0(2, -3, 18)$

Solution for the Exercise 5

1. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla f(1,1,1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow$ Tangent plane : $2(x-1) + 2(y-1) + 2(z-1) = 0 \Rightarrow x + y + z = 3$;

   (b) Normal line: $x = 1 + 2t,\ y = 1 + 2t,\ z = 1 + 2t$

2. (a) $\nabla f = -2x\mathbf{i} + 2\mathbf{k} \Rightarrow \nabla f(2,0,2) = -4\mathbf{i} + 2\mathbf{k} \Rightarrow$ Tangent plane:$-4(x-2) + 2(z-2) = 0 \Rightarrow -4x + 2z + 4 = 0 \Rightarrow -2x + z + 2 = 0$;

   (b) Normal line: $x = 2 - 4t,\ y = 0,\ z = 2 + 2t$

3. (a) $\nabla f = (-\pi \sin \pi x - 2xy + ze^{xz})\mathbf{i} + (-x^2 + z)\mathbf{j} + (xe^{xz} + y)\mathbf{k} \Rightarrow$ $\nabla f(0,1,2) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane: $2(x-0) + 2(y-1) + 1(z-2) = 0 \Rightarrow 2x + 2y + z - 4 = 0$;

   (b) Normal line: $x = 2t,\ y = 1 + 2t,\ z = 2 + t$

4. (a) $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k}$ for all points $\Rightarrow \nabla f(0,1,0) = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane: $1(x-0) + 1(y-1) + 1(z-0) = 0 \Rightarrow x + y + z - 1 = 0$;

   (b) Normal line: $x = t,\ y = 1 + t,\ z = t$

5. (a) $\nabla f = (2x - 2y - 1)\mathbf{i} + (2y - 2x + 3)\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(2,-3,18) = 9\mathbf{i} - 7\mathbf{j} - \mathbf{k} \Rightarrow$ Tangent plane: $9(x-2) - 7(y+3) - 1(z-18) = 0 \Rightarrow 9x - 7y - z = 21$;

   (b) Normal line: $x = 2 + 9t,\ y = -3 - 7t\ z = 18 - t$

Exercise 6.

Find an equation for the plane that is tangent to the given surface at the given point.

1. $z = \ln(x^2 + y^2), \quad (1, 0, 0)$
2. $z = e^{-(x^2+y^2)}, \quad (0, 0, 1)$
3. $z = 4x^2 + y^2, \quad (1, 1, 5)$

Solution for the Exercise 5

1. $z = f(x,y) = \ln(x^2+y^2) \Rightarrow f_x(x,y) = \frac{2x}{x^2+y^2}$ and $f_y(x,y) = \frac{2y}{x^2+y^2} \Rightarrow f_x(1,0) = 2$ and $f_y(1,0) = 0 \Rightarrow$ the tangent plane at $(1,0,0)$ is $2(x-1) - z = 0$ or $2x - z - 2 = 0$
2. $z = f(x,y) = e^{-(x^2+y^2)} \Rightarrow f_x(x,y) = -2xe^{-(x^2+y^2)}$ and $f_y(x,y) = -2ye^{-(x^2+y^2)} \Rightarrow f_x(0,0) = 0$ and $f_y(0,0) = 0 \Rightarrow$ the tangent plane at $(0,0,1)$ is $z - 1 = 0$ or $z = 1$
3. $z = f(x,y) = 4x^2 + y^2 \Rightarrow f_x(x,y) = 8x$ and $f_y(x,y) = 2y \Rightarrow f_x(1,1) = 8$ and $f_y(1,1) = 2 \Rightarrow$ the tangent plane at $(1,1,5)$ is $8(x-1) + 2(y-1) - (z-5) = 0$ or $8x + 2y - z - 5 = 0$

Exercise 7.

Find parametric equations for the line tangent to the curve of intersection of the surfaces at the given point.

1. *Surfaces :* $xyz = 1, \quad x^2 + 2y^2 + 3z^2 = 6$
   Point : $(1, 1, 1)$
2. *Surfaces :* $x + y^2 + z = 2, \quad y = 1$
   Point : $(1/2, 1, 1/2)$
3. *Surfaces :* $x^2 + y^2 = 4, \quad x^2 + y^2 - z = 0$
   Point : $(\sqrt{2}, \sqrt{2}, 4)$

Solution for the Exercise 7

1. $\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \Rightarrow \nabla f(1,1,1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$;
$\nabla g = 2x\mathbf{i} + 4y\mathbf{j} + 6z\mathbf{k} \Rightarrow \nabla g(1,1,1) = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$;
$\Rightarrow v = \nabla f \times \nabla g \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \Rightarrow$ Tangent line: $x = 1 + 2t,\ y = 1 - 4t,$
$z = 1 + 2t$
2. $\nabla \mathbf{f} = \mathbf{i} + 2y\mathbf{j} + \mathbf{k} \Rightarrow \nabla f(\frac{1}{2}, 1, \frac{1}{2}) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\nabla g = \mathbf{j}$ for all points;
$\mathbf{v} = \nabla \mathbf{f} \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{k} \Rightarrow$ Tangent line: $x = \frac{1}{2} - t,\ y = 1,\ z = \frac{1}{2} + t$
3. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} \Rightarrow \nabla f(\sqrt{2}, \sqrt{2}, 4) = 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j}$;
$\nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla g(\sqrt{2}, \sqrt{2}, 4) = 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} - \mathbf{k}$;
$\mathbf{v} = \nabla f \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 2\sqrt{2} & -1 \end{vmatrix} = -2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} \Rightarrow$ Tangent line:
$x = \sqrt{2} - 2\sqrt{2}t,\ y = \sqrt{2} + 2\sqrt{2}t,\ z = 4$

Estimating Change

Exercise 8.

1. *By about how much will*

$$f(x,y,z) = \ln\sqrt{x^2+y^2+z^2}$$

   change if the point $P(x,y,z)$ *moves from* $P_0(3,4,12)$ *a distance of* $ds = 0.1$ *unit in the direction of* $3\mathrm{i} + 6\mathrm{j} - 2\mathrm{k}$*?*

2. *By about how much will*

$$f(x,y,z) = e^x \cos yz$$

   change as the point $P(x,y,z)$ *moves from the origin a distance of* $ds = 0.1$ *unit in the direction of* $2\mathrm{i} + 2\mathrm{j} - 2\mathrm{k}$*?*

Solution for the Exercise 8

1. $\nabla \mathbf{f} = \left(\frac{x}{x^2+y^2+z^2}\right)\mathbf{i} + \left(\frac{y}{x^2+y^2+z^2}\right)\mathbf{j} + \left(\frac{z}{x^2+y^2+z^2}\right)\mathbf{k} \Rightarrow \nabla \mathbf{f}(3,4,12) = \frac{3}{169}\mathbf{i} + \frac{4}{169}\mathbf{j} + \frac{12}{169}\mathbf{k}$;
   $u = \frac{v}{|v|} = \frac{3\mathbf{i}+6\mathbf{j}-2\mathbf{k}}{\sqrt{3^2+6^2+(-2)^2}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k} \Rightarrow \nabla f \cdot u = \frac{9}{1183}$ and
   $df = (\nabla f \cdot u)ds = \left(\frac{9}{1183}\right)(0.1) \approx 0.0008$
2. $\nabla f = (e^x \cos yz)\mathbf{i} - (ze^x \sin yz)\mathbf{j} - (ye^x \sin yz)\mathbf{k} \Rightarrow \nabla f(0,0,0) = \mathbf{i}$;
   $\mathbf{u} = \frac{v}{|v|} = \frac{2\mathbf{i}+2\mathbf{j}-2\mathbf{k}}{\sqrt{2^2+2^2+(-2)^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow \nabla f \cdot \mathbf{u} = \frac{1}{\sqrt{3}}$ and
   $df = (\nabla \mathbf{f} \cdot \mathbf{u})ds = \frac{1}{\sqrt{3}}(0.1) \approx 0.0577$

Exercise 9.

1. *By about how much will*

$$g(x, y, z) = x + x\cos z - y\sin z + y$$

change if the point $P(x, y, z)$ *moves from* $P_0(2, -1, 0)$ *a distance of* $ds = 0.2$ *unit toward the point* $P_1(0, 1, 2)$*?*

2. *By about how much will*

$$h(x, y, z) = \cos(\pi xy) + xz^2$$

change if the point $P(x, y, z)$ *moves from* $P_0(-1, -1, -1)$ *a distance of* $ds = 0.1$ *unit toward the origin?*

Solution for the Exercise 9

1. $\nabla g = (1+\cos z)\mathbf{i} + (1-\sin z)\mathbf{j} + (-x\sin z - y\cos z)\mathbf{k} \Rightarrow$
   $\nabla g(2,-1,0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$; $\mathbf{A} = \overrightarrow{P_0P_1} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow u = \frac{v}{|v|} =$
   $\frac{-2\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{(-2)^2+2^2+2^2}} = -\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow \nabla g \cdot \mathbf{u} = 0$ and
   $dg = (\nabla g \cdot \mathbf{u})ds = (0)(0.2) = 0$
2. $\nabla h = \left[-\pi y \sin(\pi xy) + z^2\right]\mathbf{i} - [\pi x \sin(\pi xy)]\mathbf{j} + 2xz\mathbf{k} \Rightarrow$
   $\nabla h(-1,-1,-1) = (\pi\sin\pi + 1)\mathbf{i} + (\pi\sin\pi)\mathbf{j} + 2\mathbf{k} = \mathbf{i} + 2\mathbf{k}$;
   $\mathbf{v} = \overrightarrow{P_0P_1} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ where $P_1 = (0,0,0) \Rightarrow u = \frac{v}{|v|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} =$
   $\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow \nabla h \cdot \mathbf{u} = \frac{3}{\sqrt{3}} = \sqrt{3}$ and
   $dh = (\nabla h \cdot \mathbf{u})ds = \sqrt{3}(0.1) \approx 0.1732$

Exercise 10.

Suppose that the Celsius temperature at the point (x, y) in the xy-plane is

$$T(x, y) = x \sin 2y$$

and that distance in the xy-plane is measured in meters. A particle is moving clockwise around the circle of radius 1 m centered at the origin at the constant rate of 2 m/sec.

1. *How fast is the temperature experienced by the particle changing in degrees Celsius per meter at the point $P(1/\sqrt{2}, \sqrt{3}/2)$?*
2. *How fast is the temperature experienced by the particle changing in degrees Celsius per second at P?*

Solution for the Exercise 10

1. The unit tangent vector at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in the direction of motion is $u = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$;
$\nabla T = (\sin 2y)\mathbf{i} + (2x\cos 2y)\mathbf{j} \Rightarrow \nabla T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = (\sin\sqrt{3})\mathbf{i} + (\cos\sqrt{3})\mathbf{j} \Rightarrow$
$D_u T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \nabla T \cdot \mathbf{u} = \frac{\sqrt{3}}{2}\sin\sqrt{3} - \frac{1}{2}\cos\sqrt{3} \approx 0.935^\circ$ C/ft
2. $r(t) = (\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j} \Rightarrow v(t) = (2cos2t)\mathbf{i} - (2\sin 2t)\mathbf{j}$ and $|v| = 2$; $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$
$= \nabla T \cdot \mathbf{v} = \left(\nabla T \cdot \frac{v}{|v|}\right)|v| = (D_u T)|v|$, where $u = \frac{v}{|v|}$; at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we have $\mathbf{u} = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$ from part (a)
$\Rightarrow \frac{dT}{dt} = \left(\frac{\sqrt{3}}{2}\sin\sqrt{3} - \frac{1}{2}\cos\sqrt{3}\right)\cdot 2 = \sqrt{3}\sin\sqrt{3} - \cos\sqrt{3} \approx 1.87^\circ$ C/sec

Changing temperature along a space curve

Exercise 11.

The Celsius temperature in a region in space is given by

$$T(x, y, z) = 2x^2 - xyz.$$

A particle is moving in this region and its position at time t is given by $x = 2t^2,\ y = 3t,\ z = -t^2$, where time is measured in second and distance in meters.

1. *How fast is the temperature experienced by the particle changing in degrees Celsius per meter when the particle is at the point $P(8, 6, -4)$?*
2. *How fast is the temperature experienced by the particle changing in degrees Celsius per second at P?*

Solution for the Exercise 11

1. $\nabla T = (4x - yz)\mathbf{i} - xz\mathbf{j} - xy\mathbf{k} \Rightarrow \nabla T(8,6,-4) = 56\mathbf{i} + 32\mathbf{j} - 48\mathbf{k}$; $r(t) = 2t^2 i + 3t\mathbf{j} - t^2\mathbf{k} \Rightarrow$ the particle is at the point $P(8,6,-4)$ when $t = 2$; $v(t) = 4t\mathbf{i} + 3\mathbf{j} - 2t\mathbf{k} \Rightarrow v(2) = 8\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \Rightarrow u = \frac{v}{|v|} = \frac{8}{\sqrt{89}}\mathbf{i} + \frac{3}{\sqrt{89}}\mathbf{j} - \frac{4}{\sqrt{89}}\mathbf{k} \Rightarrow D_u T(8,6,-4) = \nabla T \cdot u = \frac{8}{\sqrt{89}}[56 \cdot 8 + 32 \cdot 3 - 48 \cdot (-4)] = \frac{736}{\sqrt{89}}{}^{\circ}\mathrm{C/m}$
2. $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = \nabla T \cdot v = (\nabla T \cdot u)|v| \Rightarrow$ at $t = 2$, $\frac{dT}{dt} = D_u T|_{t=2} v(2) = (\frac{736}{\sqrt{89}})\sqrt{89} = 736^{\circ}\mathrm{C/sec}$

Exercise 12.

1. *How do you find the tangent plane and normal line at a point on a level surface of a differentiable function $f(x, y, z)$? Give an example.*
2. *Find an equation for the plane tangent to the level surface $f(x, y, z) = c$ at the point P_0. Also, find parametric equations for the line that is normal to the surface at P_0.*
   (a) $x^2 - y - 5z = 0,\ P_0(2, -1, 1)$
   (b) $x^2 + y^2 + z = 4,\ P_0(1, 1, 2)$

Exercise 13.

1. *Find an equation for the plane tangent to the surface $z = f(x, y)$ at the given point.*
   (a) $z = \ln(x^2 + y^2)$, $(0, 1, 0)$
   (b) $z = 1/(x^2 + y^2)$, $(1, 1, 1/2)$
2. *Find parametric equations for the line that is tangent to the curve of intersection of the surfaces*

$$x^2 + 2y + 2z = 4, \; y = 1$$

at the given point $(1, 1, 1/2)$.

Solution for (2.) in Exercise 13

1. $\nabla f = 2x\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \nabla f(1, 1, \frac{1}{2}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\nabla g = j$ for all points; $v = \nabla f \times \nabla g \Rightarrow v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{k} \Rightarrow$ Tangent line: $x = 1 - 2t$, $y = 1$, $z = \frac{1}{2} + 2t$

Old Questions

The tangent plane to the surface $x^2 + 2y^2 + 3z^2 = 36$ at the point $P = (1, 2, 3)$ is

Select one or more:

- [ ] a. $2x + 8y + 18z = 72$
- [ ] b. None of these
- [ ] c. $2x + 8y + 18z = -72$
- [ ] d. $2x + 8y - 18z = 72$

Your answer is incorrect.

The correct answer is: $2x + 8y + 18z = 72$

Old Questions

For the given function $f(x, y) = \sqrt{16 - x^2 - y^2}$, the point $(0, 0)$ is on the level curve obtained by setting

Select one or more:

- [x] $f(x, y) = 0.$
- [ ] $f(x, y) = 2.$
- [ ] $f(x, y) = 3.$
- [ ] $f(x, y) = 4.$
- [ ] None of the other options.

The correct answer is: $f(x, y) = 4.$

Old Questions

Let $f(x, y) = \sqrt{3x^2 + 2y^2}$. Then the equation of the level curve passing through the point $(2, 2\sqrt{3})$ is

Select one or more:

- [x] $\frac{x^2}{10} + \frac{y^2}{15} = 1$
- [ ] $\frac{x^2}{6} + \frac{y^2}{9} = 1$
- [ ] $\frac{x^2}{4} + \frac{y^2}{6} = 1$
- [ ] $\frac{x^2}{12} + \frac{y^2}{18} = 1$
- [ ] None of the other options.

The correct answer is: $\frac{x^2}{12} + \frac{y^2}{18} = 1$

Old Questions

Equation of the level surface of $f(x, y, z) = \sqrt{x - y} - \log z$ passing through the point $(6, -10, e^2)$ is

Select one or more:

- [x] $f(x, y, z) = -1$
- [ ] $f(x, y, z) = 2$
- [ ] $f(x, y, z) = -2$
- [ ] $f(x, y, z) = 1$
- [ ] None of the other options.

The correct answer is: $f(x, y, z) = 2$

References

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2. R. Courant and F.John, Introduction to calculus and analysis, Volume II, Springer-Verlag.
3. N. Piskunov, Differential and Integral Calculus, Vol I & II (Translated by George Yankovsky).
4. E. Kreyszig, Advanced Engineering Mathematics, Wiley Publishers.